

On Rotating and Oscillating Four-Spin Strings in $AdS_5 \times S^5$

Kamal L. Panigrahi

*Department of Physics and Meteorology,
Indian Institute of Technology Kharagpur,
Kharagpur-721302, INDIA,
and*

*The Abdus Salam International Centre for Theoretical Physics,
Strada Costiera 11, Trieste, ITALY
E-mail: panigrahi@phy.iitkgp.ernet.in*

Pabitra M. Pradhan

*Department of Physics and Meteorology,
Indian Institute of Technology Kharagpur,
Kharagpur-721 302, INDIA
E-mail: ppabitra@phy.iitkgp.ernet.in*

ABSTRACT: We study rigidly rotating strings in $AdS_5 \times S^5$ background with one spin along AdS_5 and three angular momenta along S^5 . We find dispersion relations among various charges and interpret them as giant magnon and spiky string solutions in various limits. Further we present an example of oscillating string which oscillates in the radial direction of the AdS_5 and at the same time rotates in S^5 .

KEYWORDS: AdS-CFT Correspondence, Bosonic Strings.

Contents

1. Introduction	1
2. Rotating Strings with Four Spin in $AdS_5 \times S^5$	2
3. Case I	5
3.1 Giant Magnon Solutions	6
3.2 Spiky String Solutions	6
4. Case II	7
4.1 Giant Magnon and Spiky String solutions	7
5. Oscillating in S^5 with ρ fixed in AdS	8
6. Conclusions	10

1. Introduction

Recent advances on both the string and the gauge theory sides of the AdS/CFT correspondence [1], [2], [3] has added much of our understanding to the gauge/gravity duality. The duality aims at establishing the equivalence between the spectrum of anomalous dimensions of gauge invariant operators in the gauge theory and the energy spectrum of the string theory states. In case of AdS_5/CFT_4 duality, it is possible to see how certain simple string states actually appear as field theory operators [4] [5]. This state/operator duality becomes tractable in the strong coupling limit or the so called semiclassical limit. In this limit the appearance of integrable structures on both sides of the duality makes it simple to study and to make further predictions regarding the correspondence. The integrability arises as a quantum symmetry of operator mixing in CFT side [6], [7] and as a classical symmetry on the string world-sheet in AdS space[8]. More precisely, the integrability has improved the understanding of the equivalence between the Bethe equation for the spin chain and the corresponding classical realization of Bethe equation for the classical $AdS_5 \times S^5$ string sigma model, see e.g. [9], [10]. Study of the multi-spin rotating string solutions, e.g. in $AdS_5 \times S^5$ [11], [12], [13], [14] and the Bethe equation for the diagonalization of the integrable spin chain in the SYM theory, e.g. [6], [7], [15] have been studied in great detail. In the semiclassical limit, the study of AdS/CFT duality has triggered much of interest in recent past. In this connection, magnons, which are elementary excitations on the spin chain have been realized in [16] as dual to specific rotating semiclassical string states on $R_t \times S^2$. Soon both infinite and finite long spin chain [14], [17], [25] operators

have been mapped to different string states in various backgrounds and magnon bound states [18], [19], [20], [21], [22], [23] dual to strings on different subspaces of $AdS_5 \times S^5$ with two and three angular momenta has been found out. Infact a general class of rotating string solutions known as spiky string was found out in [24] which correspond to a higher twist operator from the field theory view point. Further it was also observed in [25] that both giant magnon and single spike solutions can be viewed as two different limits of the same rigidly rotating strings on S^2 and S^3 . In an attempt to find out new solution, a large class of multispin spiky string and giant magnon solutions have been studied, in various backgrounds including the less supersymmetric, orbifolded and non-AdS backgrounds, for example in [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39]. More recently in [40], [41], [42], [43] more general three spin giant magnons and spiky strings have been studied and interesting dispersion relation among the divergent energy and angular momenta have been obtained. In particular in [42] an interesting rotating solution has been studied with thee divergent angular momenta along the S^5 and divergent deficit angles around the sphere in $AdS_5 \times S^5$. Here, we would like to generalize the result to add another spin along AdS_5 . We also wish to generalize the four spin solutions to a class of giant magnon obtained in [22] and show the similarity of the dispersion relation to the solutions of [23] with three spin giant magnon. We wish to note that the solutions presented here correspond to rotating open strings with Neumann boundary condition.

We also describe a class of multispin oscillating string in $AdS_5 \times S^5$. The oscillating string solutions found in [5] describes a string oscillating in one plane. In [44] and [45], pulsating string solutions in AdS_5 and S^5 have been worked out separately where as in [46], rotating and oscillating strings in AdS_5 have been derived. In these papers, the solutions to the equations of motion for a general rotating and oscillating string have been discussed and the energy expression as the function of oscillation number has been derived. We wish to generalize the solutions, to describe the motion of a string which oscillates in the S^5 with a constant ρ value in AdS_5 , in subsequent section.

The rest of the paper is organized as follows. In section-2 we solve the general equation of motion for a rigidly rotating string in the background of $AdS_5 \times S^5$, with one spin along AdS_5 and three angular momenta along S^5 directions. We write down the most general solutions to the equations of motion of the fundamental string and write down the conserved quantities. In section-3 we compute all the charges and have shown a class of spiky string solutions with the help of a set of integration constants. In section-4 we present a different class of solutions with a different set of integration constants. In section-5, we present an oscillating string which oscillates in the S^5 . Finally in section-6, we conclude with some discussion and remarks.

2. Rotating Strings with Four Spin in $AdS_5 \times S^5$

The full metric for $AdS_5 \times S^5$ background is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho (d\varphi_3^2 + \cos^2 \varphi_3 d\varphi_2^2 + \sin^2 \varphi_3 d\varphi_1^2) + d\psi^2$$

$$+ \sin^2 \psi d\theta^2 + \sin^2 \psi \cos^2 \theta d\phi_1^2 + \cos^2 \psi d\phi_2^2 + \sin^2 \psi \sin^2 \theta d\phi_3^2, \quad (2.1)$$

where $\rho \in [0, \infty]$, $\theta, \psi \in [0, \frac{\pi}{2}]$, $\phi_1, \phi_2, \phi_3 \in [0, 2\pi]$ and $\varphi_1, \varphi_2, \varphi_3$ are co-ordinates on a three sphere. For studying the rotating string with three angular momenta along S^5 and one spin along AdS_5 , we substitute $\varphi_1 = \varphi_2 = 0, \varphi_3 = \varphi$ and ψ as a constant. The Polyakov action for the fundamental string in the background (2.1) with the above identification is given by

$$I = \frac{T}{2} \int d\tau d\sigma \left[-\cosh^2 \rho (\dot{t}^2 - t'^2) + \dot{\rho}^2 - \rho'^2 + \sinh^2 \rho (\dot{\varphi}^2 - \varphi'^2) + \sin^2 \psi (\dot{\theta}^2 - \theta'^2) \right. \\ \left. + \sin^2 \psi \cos^2 \theta (\dot{\phi}_1^2 - \phi_1'^2) + \cos^2 \psi (\dot{\phi}_2^2 - \phi_2'^2) + \sin^2 \psi \sin^2 \theta (\dot{\phi}_3^2 - \phi_3'^2) \right], \quad (2.2)$$

where the dot and prime denote the derivatives with respect to τ and σ respectively and the string tension T is a function of 't Hooft coupling λ as $T = \frac{\sqrt{\lambda}}{2\pi}$. We choose the following ansatz for the rotating string¹

$$t = \tau + f_1(y), \quad \rho = \rho(y), \quad \psi \in \left(0, \frac{\pi}{2}\right), \quad \theta = \theta(y), \quad \varphi = \omega \left(\tau + f_2(y)\right), \\ \phi_1 = \omega_1 \tau + g_1(y), \quad \phi_2 = \omega_2 \tau + g_2(y), \quad \phi_3 = \omega_3 \tau + g_3(y), \quad (2.3)$$

where y is a function of world sheet coordinates, $y = a\sigma - b\tau$.

From the equations of motion for t , φ , ϕ_1 , ϕ_2 and ϕ_3 , we get the expressions for f_1 , f_2 , g_1 , g_2 and g_3 which are

$$f_{1y} = \frac{1}{a^2 - b^2} \left(\frac{A_1}{\cosh^2 \rho} - b \right), \\ f_{2y} = \frac{1}{a^2 - b^2} \left(\frac{A_2}{\sinh^2 \rho} - b \right), \\ g_{1y} = \frac{1}{a^2 - b^2} \left(\frac{B_1}{\sin^2 \psi \cos^2 \theta} - b\omega_1 \right), \\ g_{2y} = \frac{1}{a^2 - b^2} \left(\frac{B_2}{\cos^2 \psi} - b\omega_2 \right), \\ g_{3y} = \frac{1}{a^2 - b^2} \left(\frac{B_3}{\sin^2 \psi \sin^2 \theta} - b\omega_3 \right), \quad (2.4)$$

where $f_y = \frac{\partial f}{\partial y}$, $g_y = \frac{\partial g}{\partial y}$ and A_1 , A_2 , B_1 , B_2 and B_3 are integration constants. Using above (2.4) values we can get the equation of motion for ρ and θ which are

$$\rho_{yy} = -\frac{1}{(a^2 - b^2)^2} \cosh \rho \sinh \rho \left(\frac{A_1^2}{\cosh^4 \rho} - \frac{A_2^2 \omega^2}{\sinh^4 \rho} - a^2 + a^2 \omega^2 \right), \\ \theta_{yy} = \frac{1}{(a^2 - b^2)^2} \cos \theta \sin \theta \left(\frac{B_3^2}{\sin^4 \psi \sin^4 \theta} - \frac{B_1^2}{\sin^4 \psi \cos^4 \theta} + a^2 \omega_1^2 - a^2 \omega_3^2 \right), \quad (2.5)$$

¹we describe the open string solutions with Neumann boundary conditions.

where $\rho_{yy} = \frac{\partial^2 \rho}{\partial y^2}$ etc. Now, the first Virasoro constraint $T_{\tau\sigma} = 0$ gives

$$\begin{aligned}\rho_y^2 + \sin^2 \psi \theta_y^2 &= \cosh^2 \rho (f_{1y}^2 - \frac{1}{b} f_{1y}) - \omega^2 \sinh^2 \rho (f_{2y}^2 - \frac{1}{b} f_{2y}) \\ &\quad - \sin^2 \psi \cos^2 \theta (g_{1y}^2 - \frac{\omega_1}{b} g_{1y}) - \cos^2 \psi (g_{2y}^2 - \frac{\omega_2}{b} g_{2y}) \\ &\quad - \sin^2 \psi \sin^2 \theta (g_{3y}^2 - \frac{\omega_3}{b} g_{3y}),\end{aligned}\tag{2.6}$$

and the second virasoro constraint $T_{\tau\tau} + T_{\sigma\sigma} = 0$ gives

$$\begin{aligned}\rho_y^2 + \sin^2 \psi \theta_y^2 &= \cosh^2 \rho \left(f_{1y}^2 + \frac{1 - 2bf_{1y}}{a^2 + b^2} \right) - \omega^2 \sinh^2 \rho \left(f_{2y}^2 + \frac{1 - 2bf_{2y}}{a^2 + b^2} \right) \\ &\quad - \sin^2 \psi \cos^2 \theta \left(g_{1y}^2 + \frac{\omega_1^2 - 2b\omega_1 g_{1y}}{a^2 + b^2} \right) - \cos^2 \psi \left(g_{2y}^2 - \frac{\omega_2^2 - 2b\omega_2 g_{2y}}{a^2 + b^2} \right) \\ &\quad - \sin^2 \psi \sin^2 \theta \left(g_{3y}^2 + \frac{\omega_3^2 - 2b\omega_3 g_{3y}}{a^2 + b^2} \right).\end{aligned}\tag{2.7}$$

The conserved quantities are

$$\begin{aligned}E &= T \int d\sigma \cosh^2 \rho \dot{t}, \\ S &= T \int d\sigma \sinh^2 \rho \dot{\phi}, \\ J_1 &= T \int d\sigma \sin^2 \psi \cos^2 \theta \dot{\phi}_1, \\ J_2 &= T \int d\sigma \cos^2 \psi \dot{\phi}_2, \\ J_3 &= T \int d\sigma \sin^2 \psi \sin^2 \theta \dot{\phi}_3.\end{aligned}\tag{2.8}$$

Deficit angle is defined as $\Delta\phi = \int \frac{\partial\phi}{\partial y} dy$. Now, we define $S_1 = \frac{S}{\omega}$, $J_1 = \frac{J_1}{\sin^2 \psi}$, $J_2 = \frac{J_2}{\cos \psi \sin \psi}$, $J_3 = \frac{J_3}{\sin^2 \psi}$ and $\Delta\phi_2 = \frac{\Delta\phi_2}{\tan \psi}$. From the Virasoro constraints (2.6) and (2.7), we get the following relation among the integration constants

$$A_1 - A_2 \omega^2 - B_1 \omega_1 - B_2 \omega_2 - B_3 \omega_3 = 0.\tag{2.9}$$

For the convenience of our solution, the arbitrary parameters A_1 and A_2 that characterize the time and angle coordinates of string in AdS_3 are chosen as $A_1 = b$ and $A_2 = 0$. Using this in the equation of motion of ρ (2.5) and integrating it once with a zero integration constant we get

$$\rho_y^2 = \frac{1}{(a^2 - b^2)^2} \sinh^2 \rho \left[a^2 - a^2 \omega^2 - \frac{b^2}{\cosh^2 \rho} \right].\tag{2.10}$$

Subtracting the above equation (2.10) from the second Virasoro constraint (2.7), we get

$$\theta_y^2 + \frac{c_1^2}{\cos^2 \theta} + c_2^2 + \frac{c_3^2}{\sin^2 \theta} + v_1^2 \cos^2 \theta + v_2^2 + v_3^2 \sin^2 \theta - \kappa^2 = 0,\tag{2.11}$$

where

$$\begin{aligned}
c_1 &= \frac{B_1}{(a^2 - b^2) \sin^2 \psi}, & c_2 &= \frac{B_2}{(a^2 - b^2) \sin \psi \cos \psi}, & c_3 &= \frac{B_3}{(a^2 - b^2) \sin^2 \psi}, \\
v_1 &= \frac{a\omega_1}{a^2 - b^2}, & v_2 &= \frac{a\omega_2 \cos \psi}{a^2 - b^2 \sin \psi}, & v_3 &= \frac{a\omega_3}{a^2 - b^2}, \\
\kappa^2 &= \frac{a^2 + b^2}{(a^2 - b^2)^2} \frac{1}{\sin^2 \psi}.
\end{aligned} \tag{2.12}$$

For concrete realization of the rotating string solutions presented above as the giant magnon and single spike solutinos, in what follows, we wish to study two different sets of integration constants values as follows

$$\text{Case I : } B_1 \neq B_2 \neq B_3 \neq 0, \tag{2.13}$$

$$\text{Case II : } B_1 = B_2 = 0. \tag{2.14}$$

3. Case I

Now, we substitute $\xi = \cos 2\theta$ in the equation (2.11) and rewrite it

$$\begin{aligned}
\xi_y^2 &= 2(v_1^2 - v_3^2)\xi^3 + 2(v_1^2 + v_3^2 - 2m)\xi^2 + 2(v_3^2 - v_1^2 + 4(c_1^2 - c_3^2))\xi \\
&\quad - 2(v_1^2 + v_3^2) - 8(c_1^2 + c_3^2) + 4m,
\end{aligned} \tag{3.1}$$

where $m = \kappa^2 - v_2^2 - c_2^2$. In order to get the solution, we choose $c_1^2 = \frac{1}{4}(m - v_1^2)$ and $c_3^2 = \frac{1}{4}(m - v_3^2)$. Using these values in (3.1), we get

$$dy = \frac{1}{\sqrt{2(v_3^2 - v_2^2)}} \frac{d\xi}{\xi \sqrt{\xi_0 - \xi}}, \tag{3.2}$$

where $\xi_0 = \frac{v_1^2 + v_3^2 - 2m}{v_3^2 - v_1^2}$ and $\xi_0 \in (0, 1)$. Now, the conserved quantities including that of the angle differences between the end point of the string can be rewritten as

$$\begin{aligned}
E &= \frac{T}{a} \int dy + \frac{aT}{a^2 - b^2} \int dy \sinh^2 \rho, \\
S_1 &= \frac{aT}{a^2 - b^2} \int dy \sinh^2 \rho, \\
J_1 &= \left(\frac{1}{2}av_1 - bc_1\right)(E - S_1) + \frac{1}{2} \frac{Tv_1}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}}, \\
J_2 &= (av_2 - bc_2)(E - S_1), \\
J_3 &= \left(\frac{1}{2}av_3 - bc_3\right)(E - S_1) - \frac{1}{2} \frac{Tv_3}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}}, \\
T\Delta\phi_1 &= (2ac_1 - bv_1)(E - S_1) - \frac{2c_1T}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{1}{\xi + 1} \frac{d\xi}{\sqrt{\xi_0 - \xi}},
\end{aligned}$$

$$T\Delta\phi_2 = (ac_2 - bv_2)(E - S_1),$$

$$T\Delta\phi_3 = (2ac_3 - bv_3)(E - S_1) - \frac{2c_3T}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{1}{\xi - 1} \frac{d\xi}{\sqrt{\xi_0 - \xi}}. \quad (3.3)$$

3.1 Giant Magnon Solutions

Here we choose constants values in such a way that we get finite deficit angles and large energy and angular momenta. Thus

$$2a(c_1 + c_3) = b(v_1 + v_3), \quad ac_2 = bv_2, \quad ac_1 \neq bv_1, \quad ac_3 \neq bv_3. \quad (3.4)$$

We take $J = J_1 + J_3$ and $\Delta\phi = \Delta\phi_1 + \Delta\phi_3$, so

$$\begin{aligned} J &= \frac{1}{2}a(v_1 + v_3)\left(1 - \frac{b^2}{a^2}\right)(E - S_1) + \frac{1}{2} \frac{T(v_1 - v_3)}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}}, \\ J_2 &= av_2\left(1 - \frac{b^2}{a^2}\right)(E - S_1), \quad \Delta\phi_2 = 0, \\ \Delta\phi &= \sqrt{\frac{2}{v_3^2 - v_1^2}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \left(\frac{c_3}{1 - \xi} - \frac{c_1}{1 + \xi}\right). \end{aligned} \quad (3.5)$$

With the constraint $4v_2^2 + (v_1 + v_3)^2 = \frac{4a^2}{(a^2 - b^2)^2}$, we get the magnon dispersion relation

$$\sqrt{(E - S_1)^2 - J_2^2} - J = T \sin \Delta\phi, \quad (3.6)$$

where deficit angle has a constraint $\sin \Delta\phi = \sqrt{\frac{v_3 - v_1}{v_3 + v_1}} \sqrt{\frac{\xi_0}{2}}$.

3.2 Spiky String Solutions

Similarly, to get the finite momenta and large energy and deficit angles, we choose the conditions as

$$2b(c_1 + c_3) = a(v_1 + v_3), \quad bc_2 = av_2, \quad av_1 \neq bc_1, \quad av_3 \neq bc_3. \quad (3.7)$$

So

$$\begin{aligned} J &= \frac{1}{2} \frac{T(v_1 - v_3)}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}}, \\ T\Delta\phi &= \left(\frac{a^2}{b^2} - 1\right)b(v_1 + v_3)(E - S_1) - \frac{2T}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \left(\frac{c_1}{\xi + 1} - \frac{c_3}{\xi - 1}\right), \\ T\Delta\phi_2 &= \left(\frac{a^2}{b^2} - 1\right)bv_2(E - S_1), \quad J_2 = 0. \end{aligned} \quad (3.8)$$

With the constraint $v_2^2 + (v_1 + v_3)^2 = \frac{b^2}{(a^2 - b^2)^2}$, we get the spiky dispersion relation

$$\sqrt{(E - S_1)^2 - (T\Delta\phi_2)^2} - T\Delta\phi = 2T\left(\frac{\pi}{2} - \theta_0\right), \quad (3.9)$$

where

$$\theta_0 = \frac{\pi}{2} - \frac{1}{\sqrt{2(v_3^2 - v_1^2)}} \int_0^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \left(\frac{c_1}{\xi + 1} - \frac{c_3}{\xi - 1}\right), \quad \text{and} \quad \xi_0 = \cos 2\theta_0.$$

4. Case II

Putting condition(2.14) in the equation (2.11), we get

$$\theta_y^2 = \kappa_1^2 - c_3^2 \frac{\cos^2 \theta}{\sin^2 \theta} - (v_1^2 + v_2^2) - (v_3^2 - v_1^2) \sin^2 \theta, \quad (4.1)$$

where $\kappa_1^2 = \frac{(a^2+b^2)\omega_3^2 \sin^2 \psi - b^2}{(a^2-b^2)^2 \omega_3^2 \sin^4 \psi}$ and we choose $\kappa_1^2 = v_2^2 + v_3^2$. Now, equation(4.1) can be written as

$$\theta_y^2 = \frac{v_3^2 - v_1^2}{\sin^2 \theta} (\sin^2 \theta - \sin^2 \theta_1) (\sin^2 \theta_0 - \sin^2 \theta), \quad (4.2)$$

where

$$\sin^2 \theta_0 = \frac{c_3^2}{v_3^2 - v_1^2}, \quad \sin^2 \theta_1 = \frac{\kappa_1^2 - v_1^2 - v_2^2}{v_3^2 - v_1^2}. \quad (4.3)$$

From (4.3), We get the two conditions (i) $\kappa_1^2 = v_2^2 + v_3^2$ and (ii) $c_3^2 = v_3^2 - v_1^2$. Both the conditions will give us same dy equation, only with different θ_0 value. So the conserved quantities will remain same and this will lead us to same relations. Here we study the different dispersion relations with the (i) condition only. So taking $\kappa_1^2 = v_2^2 + v_3^2$, the equation (4.2) changes to

$$dy = \frac{1}{\sqrt{v_3^2 - v_1^2}} \frac{\sin \theta}{\cos \theta} \frac{d\theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}}. \quad (4.4)$$

Thus, the conserved quantities are rewritten as

$$\begin{aligned} E &= \frac{T}{a} \int dy + \frac{aT}{a^2 - b^2} \int dy \sinh^2 \rho, \\ S_1 &= \frac{aT}{a^2 - b^2} \int dy \sinh^2 \rho, \\ J_1 &= \frac{Tv_1}{\sqrt{v_3^2 - v_1^2}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin \theta \cos \theta d\theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \\ J_2 &= av_2(E - S_1), \\ J_3 &= (av_3 - bc_3)(E - S_1) - \frac{Tv_3}{\sqrt{v_3^2 - v_1^2}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin \theta \cos \theta d\theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \\ T\Delta\phi_1 &= -bv_1(E - S_1), \\ T\Delta\phi_2 &= -bv_2(E - S_1), \\ T\Delta\phi_3 &= (ac_3 - bv_3)(E - S_1) + \frac{Tc_3}{\sqrt{v_3^2 - v_1^2}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} \frac{d\theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}}. \end{aligned} \quad (4.5)$$

4.1 Giant Magnon and Spiky String solutions

For the region of $0 \leq 1 - \omega^2 \leq \frac{b^2}{a^2}$, we can rewrite the eqn (2.10) as

$$\rho_y = \pm \frac{\sqrt{1 - \omega^2}}{a^2 - b^2} a \tanh \rho \sqrt{\sinh^2 \rho - \alpha^2}, \quad (4.6)$$

where $\alpha = \sqrt{\frac{\frac{b^2}{a^2} + \omega^2 - 1}{1 - \omega^2}}$. The solution for the (4.6) is

$$\sinh \rho = \frac{\alpha}{\cos \beta y}, \quad \text{for } -\frac{\pi}{2\beta} \leq y \leq \frac{\pi}{2\beta}, \quad (4.7)$$

where $\beta = \frac{\sqrt{b^2 + a^2(\omega^2 - 1)}}{a^2 - b^2}$. Similarly for the region of $\frac{b^2}{a^2} \leq 1 - \omega^2$, we get the solution as

$$\begin{aligned} \sinh \rho &= \frac{\alpha}{\sinh \beta y}, & \text{for } 0 \leq y < \infty, \\ &= -\frac{\alpha}{\sinh \beta y}, & \text{for } -\infty < y \leq 0, \end{aligned} \quad (4.8)$$

where $\alpha = \sqrt{\frac{1 - \omega^2 - \frac{b^2}{a^2}}{1 - \omega^2}}$ and $\beta = \frac{\sqrt{a^2(\omega^2 - 1) - b^2}}{a^2 - b^2}$. Using (4.6), (4.7) and (4.8), we can compute the deficit angle for time as

$$\Delta t = -2 \arctan \frac{\sqrt{1 - \alpha^2}}{\alpha}. \quad (4.9)$$

Here α is characterized by a time difference between the two endpoints of the open string. In order to get the dispersion relation, we choose the condition among following constants as

$$v_3 = \frac{1}{2a}(1 + bc_3). \quad (4.10)$$

Using (4.5) and (4.10), we get the giant magnon dispersion relation as

$$E - J_3 = S_1 + \frac{v_3}{v_1} J_1 + \frac{v_3}{v_2} J_2. \quad (4.11)$$

We can define $J = J_3 + \frac{v_3}{v_1} J_1$ and make the $\Delta\phi_3$ finite by choosing $ac_3 = bv_3$. From [23], we can regularize S_1 and from [18] and [19], we can write down the finite terms for $\frac{v_3}{v_2} J_2$, which will give us the dispersion relation as

$$(E - J)_{reg} = -\sqrt{S_{reg}^2 + \frac{\lambda}{\pi^2} \cos^2 \frac{\Delta t}{2}} + \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{\Delta\phi_3}{2}}. \quad (4.12)$$

Similarly, we choose the condition for spiky string as

$$v_1^2 + v_2^2 + v_3^2 = \frac{1}{b^2}(1 - a^2 c_3^2) + 2\frac{a}{b} c_3 v_3. \quad (4.13)$$

Using (4.13) and (4.5), we get the spiky string dispersion relation

$$\sqrt{(E - S_1)^2 - (T\Delta\phi_1)^2 - (T\Delta\phi_2)^2} - T\Delta\phi_3 = 2T(\frac{\pi}{2} - \theta_0). \quad (4.14)$$

5. Oscillating in S^5 with ρ fixed in AdS

Here we consider a string oscillating in the ψ direction of the S^5 , which is periodic in nature and ρ is fixed to ρ_0 in AdS . We wish to study a configuration with two spin in AdS

and two angular momenta along S^5 . For the study of this oscillating string we substitute $\varphi_3 = \frac{\pi}{4}$ and $\phi_1 = \phi_3$ in (2.1) and get the metric as

$$ds^2 = -\cosh^2 \rho_0 dt^2 + \frac{1}{2} \sinh^2 \rho_0 (d\varphi_1^2 + d\varphi_2^2) + d\psi^2 \\ + \sin^2 \psi d\theta^2 + \sin^2 \psi d\phi_1^2 + \cos^2 \psi d\phi_2^2. \quad (5.1)$$

Now we parameterize the coordinates as

$$t = \kappa\tau, \quad \varphi_i = \mu_i\tau, \quad \psi = \psi(\tau), \quad \theta = m\sigma, \quad \phi_i = \nu_i\tau, \quad (5.2)$$

where $i = 1, 2$. The Nambu-Goto action for the string in the background (5.1) with the above ansatz (5.2) is given by

$$I = m\sqrt{\lambda} \int d\tau \sin \psi \sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}. \quad (5.3)$$

The conserved quantities derived from the above action (5.3) are

$$E = \frac{m\sqrt{\lambda}\kappa \sin \psi \cosh^2 \rho_0}{\sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}}, \\ S_1 = \frac{m\sqrt{\lambda}\mu_1 \sin \psi \sinh^2 \rho_0}{2\sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}}, \\ S_2 = \frac{m\sqrt{\lambda}\mu_2 \sin \psi \sinh^2 \rho_0}{2\sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}}, \\ J_1 = \frac{m\sqrt{\lambda}\nu_1 \sin^3 \psi}{\sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}}, \\ J_2 = \frac{m\sqrt{\lambda}\nu_2 \sin \psi \cos^2 \psi}{\sqrt{\kappa^2 \cosh^2 \rho_0 - \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 - \dot{\psi}^2 - \nu_1^2 \sin^2 \psi - \nu_2^2 \cos^2 \psi}}. \quad (5.4)$$

From the above equation (5.4), we get a relation among the conserved charges

$$\frac{E}{\kappa} - \frac{S_1}{\mu_1} - \frac{S_2}{\mu_2} = \frac{J_1}{\nu_1} + \frac{J_2}{\nu_2}. \quad (5.5)$$

This (5.5) is the dispersion relation for a string which is oscillating in the S^5 from a minimum value of ψ_{min} to a maximum $(\frac{\pi}{2} - \psi_{min})$ and at the same time it rotates with two angular momenta. Now solving the Euler-Lagrangian equation for t with appropriate integration constant we get the equation of motion which is

$$\dot{\psi}^2 - \kappa^2 \cosh^2 \rho_0 + \frac{1}{2}(\mu_1^2 + \mu_2^2) \sinh^2 \rho_0 + m^2 \cosh^2 \rho_0 \sin^2 \psi \\ + \nu_1^2 \sin^2 \psi + \nu_2^2 \cos^2 \psi = 0. \quad (5.6)$$

The above equation (5.6) gives us the potential energy $V(\psi)$ which gives the oscillation number for the strings. If we put $\mu_i = \nu_i = 0$ in the above equation (5.6), it changes to

$$\dot{\psi}^2 = \kappa^2 \cosh^2 \rho_0 - m^2 \cosh^2 \rho_0 \sin^2 \psi, \quad (5.7)$$

which is similar to the conformal gauge condition obtained in [45]. With $\mu_i = \nu_i = 0$, the conserved quantities $S_{i=1,2}$ and $J_{i=1,2}$ become zero. The energy and oscillation number imply

$$\begin{aligned} \mathcal{E} &= \frac{E}{\sqrt{\lambda}} = \kappa \cosh \rho_0, \\ \mathcal{N} &= \frac{N}{\sqrt{\lambda}} = \frac{1}{2\pi} \oint d\psi \dot{\psi} = \frac{2n}{\pi} \left[\left(\frac{\gamma^2}{n^2} - 1 \right) \mathbb{K} \left(\frac{\gamma^2}{n^2} \right) + \mathbb{E} \left(\frac{\gamma^2}{n^2} \right) \right], \end{aligned} \quad (5.8)$$

where \mathbb{K} and \mathbb{E} are the usual elliptical functions, $\gamma = \kappa \cosh \rho_0$ and $n = m \cosh \rho_0$. For small γ and \mathcal{N} , we get the classical energy as

$$\mathcal{E} = \sqrt{2n\mathcal{N}} + O\left(\mathcal{N}^{\frac{3}{2}}\right), \quad (5.9)$$

which is similar to the relation got in [45]. If we consider $\rho_0 = 0$, then we get back the exact energy for the string oscillating in the $R \times S^2$. It will be interesting to find out solutions of open string that oscillates in AdS_5 and at the same time rotates along S^5 .

6. Conclusions

In this paper, we have found two classes of giant magnon and spiky string solutions in $\text{AdS}_5 \times S^5$ background with one spin along AdS_5 and three angular momenta in S^5 . We have calculated the divergent energies and angular momenta and have found out new solutions corresponding to giant magnon and spiky strings. The dispersion relations we got in (3.6) and (3.9) are the generalized solutions of [42] to include a spin along AdS_5 . Similarly, by putting $S_1 = 0$ in the (4.11) we get the relation obtained in [22]. Though, in [22] all the three angular momenta are infinite, in our case one of the momenta (J_1) is finite. The dispersion relation found in the (4.12) is similar to [23], where we combine the two momenta to get the energy of a superposition of two magnon bound states. The last dispersion relation in (4.14) presents a new class of spiky string solution. However, with a different constraints of integration constants we can get a dispersion relation same as obtained in (3.9). There are further questions that can be addressed. First of all it will be interesting to look for solutions with divergent energy and multiple angular momenta in the β deformed $\text{AdS}_5 \times S^5$. However, the presence of various background fluxes makes the study more difficult, nevertheless worth exploring. According to AdS/CFT duality these states would correspond to some gauge theory operators, though the exact nature of the operators are not known, but it will most likely fall into the class of operators that correspond to the known multispin solutions on the $\text{AdS}_5 \times S^5$. It will be interesting to look at the operator dual of such states in detail. For example the oscillating and rotating string solutions present in the last section might correspond to certain highly excited sigma

model operators. These in turn must have something to do with the higher spin states and it will be exciting to study the nature of these states and operators.

Acknowledgements: KLP would like to thank the Abdus Salam I.C.T.P, Trieste for hospitality under Associate Scheme, where a part of this work was completed.

References

- [1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [[hep-th/9711200](#)].
- [2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)].
- [3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [4] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” *JHEP* **0204**, 013 (2002) [[hep-th/0202021](#)].
- [5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A Semiclassical limit of the gauge / string correspondence,” *Nucl. Phys. B* **636**, 99 (2002) [[hep-th/0204051](#)].
- [6] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” *JHEP* **0303**, 013 (2003) [[hep-th/0212208](#)].
- [7] N. Beisert and M. Staudacher, “The N=4 SYM integrable super spin chain,” *Nucl. Phys. B* **670**, 439 (2003) [[hep-th/0307042](#)].
- [8] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” *Phys. Rev. D* **69**, 046002 (2004) [[hep-th/0305116](#)].
- [9] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” *JHEP* **0405**, 024 (2004) [[hep-th/0402207](#)].
- [10] K. Zarembo, “Semiclassical Bethe Ansatz and AdS/CFT,” *Comptes Rendus Physique* **5**, 1081 (2004) [*Fortsch. Phys.* **53**, 647 (2005)] [[hep-th/0411191](#)].
- [11] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S**5,” *JHEP* **0206**, 007 (2002) [[hep-th/0204226](#)].
- [12] S. Frolov and A. A. Tseytlin, “Multispin string solutions in AdS(5) x S**5,” *Nucl. Phys. B* **668**, 77 (2003) [[hep-th/0304255](#)].
- [13] S. Frolov and A. A. Tseytlin, “Quantizing three spin string solution in AdS(5) x S**5,” *JHEP* **0307**, 016 (2003) [[hep-th/0306130](#)].
- [14] A. A. Tseytlin, “Spinning strings and AdS / CFT duality,” In **Shifman, M. (ed.) et al.: From fields to strings*, vol. 2* 1648-1707 [[hep-th/0311139](#)].
- [15] N. Beisert, “The Dilatation operator of N=4 super Yang-Mills theory and integrability,” *Phys. Rept.* **405**, 1 (2005) [[hep-th/0407277](#)].
- [16] D. M. Hofman and J. M. Maldacena, “Giant Magnons,” *J. Phys. A* **39**, 13095 (2006) [[hep-th/0604135](#)].
- [17] A. A. Tseytlin, “Semiclassical strings and AdS/CFT,” [[hep-th/0409296](#)].

- [18] G. Arutyunov, S. Frolov and M. Zamaklar, “Finite-size Effects from Giant Magnons,” Nucl. Phys. B **778**, 1 (2007) [hep-th/0606126].
- [19] J. A. Minahan, A. Tirziu and A. A. Tseytlin, “Infinite spin limit of semiclassical string states,” JHEP **0608**, 049 (2006) [hep-th/0606145].
- [20] N. Dorey, “Magnon Bound States and the AdS/CFT Correspondence,” J. Phys. A **39**, 13119 (2006) [hep-th/0604175].
- [21] H. -Y. Chen, N. Dorey and K. Okamura, “Dyonic giant magnons,” JHEP **0609**, 024 (2006) [hep-th/0605155].
- [22] M. Kruczenski, J. Russo and A. A. Tseytlin, “Spiky strings and giant magnons on S^{*5} ,” JHEP **0610**, 002 (2006) [hep-th/0607044].
- [23] S. Ryang, “Three-spin giant magnons in $AdS(5) \times S^{*5}$,” JHEP **0612**, 043 (2006) [hep-th/0610037].
- [24] M. Kruczenski, “Spiky strings and single trace operators in gauge theories,” JHEP **0508**, 014 (2005) [hep-th/0410226].
- [25] R. Ishizeki and M. Kruczenski, “Single spike solutions for strings on S^{*2} and S^{*3} ,” Phys. Rev. D **76**, 126006 (2007) [arXiv:0705.2429 [hep-th]].
- [26] O. Lunin and J. M. Maldacena, “Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals,” JHEP **0505**, 033 (2005) [hep-th/0502086].
- [27] N. P. Bobev, H. Dimov, R. C. Rashkov, “Semiclassical strings in Lunin-Maldacena background,” [hep-th/0506063].
- [28] C. -S. Chu, G. Georgiou and V. V. Khoze, “Magnons, classical strings and beta-deformations,” JHEP **0611**, 093 (2006) [hep-th/0606220].
- [29] N. P. Bobev and R. C. Rashkov, “Multispin Giant Magnons,” Phys. Rev. D **74**, 046011 (2006) [hep-th/0607018].
- [30] J. Kluson, R. R. Nayak, K. L. Panigrahi, “Giant Magnon in NS5-brane Background,” JHEP **0704**, 099 (2007). [hep-th/0703244].
- [31] B. -H. Lee, R. R. Nayak, K. L. Panigrahi, C. Park, “On the giant magnon and spike solutions for strings on $AdS(3) \times S^{*3}$,” JHEP **0806**, 065 (2008). [arXiv:0804.2923 [hep-th]].
- [32] J. R. David, B. Sahoo, “Giant magnons in the D1-D5 system,” JHEP **0807**, 033 (2008). [arXiv:0804.3267 [hep-th]].
- [33] G. Grignani, T. Harmark and M. Orselli, “The $SU(2) \times SU(2)$ sector in the string dual of $N=6$ superconformal Chern-Simons theory,” Nucl. Phys. B **810**, 115 (2009) [arXiv:0806.4959 [hep-th]].
- [34] B. -H. Lee, K. L. Panigrahi, C. Park, “Spiky Strings on $AdS(4) \times CP^{*3}$,” JHEP **0811**, 066 (2008). [arXiv:0807.2559 [hep-th]].
- [35] S. Ryang, “Giant Magnon and Spike Solutions with Two Spins in $AdS(4) \times CP^{*3}$,” JHEP **0811**, 084 (2008). [arXiv:0809.5106 [hep-th]].
- [36] S. Benvenuti and E. Tonni, “Giant magnons and spiky strings on the conifold,” JHEP **0902**, 041 (2009) [arXiv:0811.0145 [hep-th]].

- [37] M. C. Abbott, I. Aniceto, “Giant Magnons in $AdS(4) \times CP^3$: Embeddings, Charges and a Hamiltonian,” *JHEP* **0904**, 136 (2009). [arXiv:0811.2423 [hep-th]].
- [38] S. Biswas and K. L. Panigrahi, “Spiky Strings on NS5-branes,” *Phys. Lett. B* **701**, 481 (2011) [arXiv:1103.6153 [hep-th]],
- [39] S. Biswas and K. L. Panigrahi, “Spiky Strings on I-brane,” arXiv:1206.2539 [hep-th].
- [40] S. Giardino and H. L. Carrion, “Classical strings in $AdS(4) \times CP(3)$ with three angular momenta,” *JHEP* **1108**, 057 (2011) [arXiv:1106.5684 [hep-th]].
- [41] K. L. Panigrahi, P. M. Pradhan and P. K. Swain, “Rotating Strings in $AdS(4) \times CP(3)$ with $B(NS)$ holonomy,” *JHEP* **1201**, 113 (2012) [arXiv:1109.2458 [hep-th]].
- [42] S. Giardino, “Divergent energy strings in $AdS_5 \times S^5$ with three angular momenta,” *JHEP* **1112**, 022 (2011) [arXiv:1110.3682 [hep-th]].
- [43] K. L. Panigrahi, P. M. Pradhan and P. K. Swain, “Three Spin Spiky Strings in β -deformed Background,” *JHEP* **1206**, 057 (2012) [arXiv:1203.3057 [hep-th]].
- [44] J. A. Minahan, “Circular semiclassical string solutions on $AdS(5) \times S(5)$,” *Nucl. Phys. B* **648**, 203 (2003) [hep-th/0209047].
- [45] M. Beccaria, G. V. Dunne, G. Macorini, A. Tirziu and A. A. Tseytlin, “Exact computation of one-loop correction to energy of pulsating strings in $AdS_5 \times S^5$,” *J. Phys. A* **44**, 015404 (2011) [arXiv:1009.2318 [hep-th]].
- [46] I. Y. Park, A. Tirziu and A. A. Tseytlin, “Semiclassical circular strings in $AdS(5)$ and ‘long’ gauge field strength operators,” *Phys. Rev. D* **71**, 126008 (2005) [hep-th/0505130].